There are two ways to prove the quotient rule. The easier way is to simply use the product rule though many would consider this "incomplete" since the natural question then becomes how do you prove the product rule. That proof goes as follows:

$$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x)(g(x))^{-1}\right]' = f'(x)(g(x))^{-1} + f(x)(-1)[g(x)]^{-2}g'(x)$$

and this can be simplified (using a common denominator) to

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

If you don't wish to use the product rule, that is to prove the quotient rule from scratch essentially, you need to use the limit definition of derivative, that is that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So applying this to $\frac{f(x)}{g(x)}$, it becomes (and you use a common denominator again)

$$\left[\frac{f(x)}{g(x)}\right]' = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h}$$

which becomes

$$\lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

and then you introduce two new terms in the numerator in order to later simplify as such

$$\lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

After this, break this into two limits (which was why you introduced those new terms) like this

$$\lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x)}{hg(x+h)g(x)} + \lim_{h \to 0} \frac{f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

Then simplify these two limits into

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \frac{1}{g(x+h)} + \lim_{h \to 0} \left[\frac{g(x) - g(x+h)}{h}\right] \left[\frac{f(x)}{g(x)g(x+h)}\right]$$

Applying limit laws you then get

$$\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0}\frac{1}{g(x+h)}\right]+\left[\lim_{h\to 0}\frac{g(x)-g(x+h)}{h}\right]\left[\lim_{h\to 0}\frac{f(x)}{g(x)g(x+h)}\right]$$

so finally applying the limit definition of derivative (noted at the beginning of this proof) and then using a common denominator once more, we obtain

$$\left[\frac{f(x)}{g(x)}\right]' = f'(x)\frac{1}{g(x)} + -g'(x)\frac{f(x)}{[g(x)]^2} = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

which is the quotient rule.